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Nontrivial application of Nielsen theory to differential systems

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Abstract

In reply to a problem of Jean Leray concerning application of the Nielsen theory to differential systems for obtaining multiplicity results, we present a nontrivial example of such an application. The emphasis is on the parameter space in order to ensure that no subdomain becomes subinvariant under the related Hammerstein solution operator. To achieve this goal, we develop a general method applicable also for ordinary differential equations with or without uniqueness as well as for upper-Carathéodory differential inclusions. We are not aware that any alternative approach can be employed, even in the single-valued case.

MSC: 34A60; 34B15; 34C25; 47H10; 54H25

Keywords: Nielsen number; Lower estimate of solutions; Nontrivial example; Three periodic solutions; No subdomain subinvariantness; Differential equations and inclusions

1. Introduction

Nielsen theory is one of the few fixed point theories dealing with more than one fixed point. It allows us to obtain a lower estimate for the number of fixed points. Its central notion, the Nielsen number, is a homotopy invariant, but the Nielsen theory is rather geometrical than topological. Nevertheless, topological fixed point theory is often referred to as the Nielsen theory.

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At the first International Congress of Mathematics held after World War II in Cambridge, MA in 1950, Jean Leray suggested the *problem* of adapting the Nielsen theory to the needs of nonlinear analysis and, in particular, *of its application to differential systems for obtaining multiplicity results*. Perhaps because of the difficulty of the problem, there have been only few related contributions (see [1,2,6,11–13,15,17–22,26–28,30]). For a detailed survey of the results, see [3,14,22].

In reply to Leray's problem, which is still far from being solved in a satisfactory way, we tried to construct a nontrivial example of the application of the Nielsen number to a planar differential system without any implemented parameters (which were present in the papers of R.F. Brown [11–13,15] and M. Fečkan [17–21]), but as observed in [16], there was a gap concerning the assumptions imposed on the related Hammerstein solution operator. This gap can be simply avoided by adding some additional restrictions (see [5]), but then the usage of the Nielsen number becomes unnecessary and the results can be obtained by an alternative approach.

On the other hand, recently we succeeded (see [6]) in constructing an example of the application of the standard single-valued Nielsen theorem to a planar system of integro-differential equations without the above mentioned drawback. In [4], we also presented a multivalued version of the result in [6].

The purpose of this article is two-fold:

- to derive a general method for application of the multivalued Nielsen theory developed in [9] (cf. [1,4,7]) for operator differential inclusions with suitable constraints, and then
- to construct, on the basis of this method, a nontrivial example of a system of differential inclusions which has at least two periodic solutions.

For this goal, we briefly sketch the basic facts related to the Nielsen number for compact multivalued maps on ANR-spaces, developed in [9]. By a metric *AR-space* (respectively *ANR-space*), we understand such a space X that, for any metric space Y, its arbitrary closed subset $S \subset Y$ and any continuous mapping $g : S \to X$, there exists a continuous extension of g onto Y (respectively onto some neighbourhood of S in Y). Roughly speaking, AR-spaces (respectively ANR-spaces) are, up to retractions and up to homeomorphic images, normed spaces (respectively open subsets of normed spaces). For more details concerning ANR-spaces, see [10,24].

In the entire text, all spaces are at least metric and all multivalued maps are always assumed to have nonempty values, i.e., $\varphi : X \multimap Y$ means that $\varphi : X \to 2^Y \setminus \{\emptyset\}$.

2. Nielsen number for compact multivalued maps

The standard Nielsen theory allows us to obtain a lower estimate for the number of fixed points. More precisely, if $f : X \to X$ is a compact (continuous) map on a (metric) ANR-space X, then a nonnegative integer $N(f)$, called the *Nielsen number* of f, is defined such that

- $N(f) \leqslant \#\mathrm{Fix}(f) := \mathrm{card}\{x \in X \mid x = f(x)\}$,
- $N(f) = N(\tilde{f})$, for any compact $\tilde{f} : X \to X$ which is *compactly homotopic* to f, i.e., there exists a compact map $h : X \times [0,1] \to X$ such that $h(\cdot, 0) = f$ and $h(\cdot, 1) = \tilde{f}$.

Given a compact $f : X \to X$ on an ANR-space X, we say that $x, y \in \mathrm{Fix}(f)$ are *Nielsen related* if there exists a path $u : [0,1] \to X$ such that $u(0) = x$, $u(1) = y$ and u, $f(u)$ are homotopic keeping the endpoints fixed. Two Nielsen non-related fixed points are indicated in Fig. 2 below. Since the Nielsen relation is an equivalence, $\mathrm{Fix}(f)$ splits into fixed point classes. Since the classes are open and f is compact, we have a finite number of fixed point classes.

If, for a Nielsen class $\mathcal{N} \subset \mathrm{Fix}(f)$, we have $\mathrm{ind}(\mathcal{N}, f) \neq 0$, i.e., if the associated fixed point index is nontrivial, then $\mathcal{N}$ is called *essential*. The *Nielsen number* $N(f)$ is then defined to be the *number of essential fixed point classes*. For more details, see, e.g., [25].

To compute $N(f)$ can be a difficult task. In the multivalued case, the situation is even more delicate, because the above definition cannot be directly generalized. Thus, we need to recall this subtle definition again. Nevertheless, in the single-valued case, both the definitions are equivalent.

Consider a multivalued map $\varphi : X \multimap X$, where

(i) X is a connected ANR-space (e.g., a connected retract of an open subset of a Banach space or its closure),
(ii) X has finitely generated Abelian fundamental group,
(iii) φ is a compact (i.e., $\overline{\varphi(X)}$ is compact) composition of an R_δ-map $p^{-1} : X \multimap \Gamma$ and a continuous (single-valued) map $q : \Gamma \to X$, namely, $\varphi = q \circ p^{-1}$, where Γ is a metric space.

Then a nonnegative integer $N(\varphi) = N(p, q)$,[2] called the *Nielsen number* for φ exists (for its definition, see [9]; cf. [7] or [5]) such that

$$N(\varphi) \leqslant \#C(\varphi), \quad \text{where} \tag{1}$$

$$\#C(\varphi) = \#C(p, q) := \mathrm{card}\bigl\{z \in \Gamma \bigm| p(z) = q(z)\bigr\} \quad \text{and} \tag{2}$$

$$N(\varphi_0) = N(\varphi_1), \tag{3}$$

for compactly homotopic maps $\varphi_0 \sim \varphi_1$.

Some remarks are in order. Condition (ii) is satisfied, e.g., for the torus $\mathbb{T}^n$ (cf. [9]) or it can be avoided if X is compact and $q = \mathrm{id}$ is the identity (cf. [1,23]).

By an *R_δ-map* $p^{-1} : X \multimap \Gamma$, we mean an upper semicontinuous (u.s.c.) one (i.e., for every open $U \subset \Gamma$, the set $\{x \in X \mid p^{-1}(x) \subset U\}$ is open in X) with R_δ-values (i.e., Y is an R_δ-set if $Y = \bigcap\{Y_n \mid n = 1, 2, \ldots\}$, where $\{Y_n\}$ is a decreasing sequence of compact AR-spaces).

Let $X \overset{p_0}{\Longleftarrow} \Gamma_0 \overset{q_0}{\to} X$ and $X \overset{p_1}{\Longleftarrow} \Gamma_1 \overset{q_1}{\to} X$ be two maps, namely, $\varphi_0 = q_0 \circ p_0^{-1}$ and $\varphi_1 = q_1 \circ p_1^{-1}$. We say that φ_0 is *homotopic* to φ_1 (written $\varphi_0 \sim \varphi_1$ or $(p_0, q_0) \sim (p_1, q_1)$) if there exists a multivalued map $X \times [0,1] \overset{p}{\leftarrow} \overline{\Gamma} \overset{q}{\to} X$ such that the following diagram is commutative:

$$\begin{tikzcd}
X \arrow[d, "k_i"'] & \Gamma_i \arrow[l, Leftarrow, "p_i"'] \arrow[r, "q_i"] \arrow[d, "f_i"] & X \\
X \times [0,1] & \overline{\Gamma} \arrow[l, Leftarrow, "p"] \arrow[ur, "q"] &
\end{tikzcd}$$

for $k_i(x) = (x, i)$, $i = 0, 1$, and $f_i : \Gamma_i \to \overline{\Gamma}$ is a homeomorphism onto $p^{-1}(X \times i)$, $i = 0, 1$, i.e., $k_0 p_0 = p f_0$, $q_0 = q f_0$, $k_1 p_1 = p f_1$ and $q_1 = q f_1$. By *compactly homotopic maps* $\varphi_0 \sim \varphi_1$, we mean that the mapping $q \circ p^{-1} : X \times [0,1] \multimap X$ in the above diagram is still compact.

[2] We should write more correctly $N_H(\varphi) = N_H(p, q)$, because it is in fact (mod H)-Nielsen number. For the sake of simplicity, we omit the index H in the sequel.

Remark 1. (Important) We have an example in [9] that, under the above assumptions (i)–(iii), the Nielsen number $N(\varphi)$ is rather a topological invariant (see (3)) for the number of essential classes of coincidences (see (1), (2)) than for the number of fixed points. On the other hand, for a compact X and $q = \mathrm{id}$, $N(\varphi)$ gives even without (ii) a lower estimate of the number of fixed points of φ (see [1]), i.e., $N(\varphi) \leqslant \#\mathrm{Fix}(\varphi)$, where

$$\#\mathrm{Fix}(\varphi) := \mathrm{card}\{x \in X \mid x \in \varphi(x)\}.$$

It is obvious that if $\varphi = p^{-1}$, i.e., if φ is an R_δ-map, then we can always put $q = \mathrm{id}$ by which $\mathrm{Fix}(\varphi) = C(\varphi)$, where $\mathrm{Fix}(\varphi) := \{x \in X \mid x \in \varphi(x)\}$, and so we have under (i)–(iii)

$$N(\varphi) \leqslant \#\mathrm{Fix}(\varphi) = \#C(\varphi).$$

In the sequel, we shall also employ the following reduction property which is proved in [2].

Lemma 1. *Let X and its closed subset Y be ANR-spaces. Assume that $f : X \to X$ is a compact map such that $f(X) \subset Y$. Denoting by $f' : Y \to Y$ the restriction of f to Y, we have*

- $\mathrm{Fix}(f') = \mathrm{Fix}(f)$,
- *the Nielsen relations coincide,*
- $\mathrm{ind}(\mathcal{N}, f') = \mathrm{ind}(\mathcal{N}, f)$, *for any Nielsen class* $\mathcal{N} \subset \mathrm{Fix}(f)$.

Thus, $N(f') = N(f)$.

3. Application of Nielsen theory to operator inclusions

The Nielsen number is now going to be used to formulate a multiplicity criterium for an abstract operator differential inclusion. At first, let us prove a technical lemma which justifies the setting of the abstract problem posed later.

Let $I \subset \mathbb{R}$ be a compact interval. Denote by $C(I, \mathbb{R}^n)$ the space of all continuous functions with values in $\mathbb{R}^n$. This space is endowed with the usual topology of uniform convergence. Consider a multivalued map $H : I \times \mathbb{R}^n \multimap \mathbb{R}^n$ which satisfies the following properties:

(a) H has nonempty, compact and convex values,
(b) $H(\cdot, x)$ is measurable, for all $x \in \mathbb{R}^n$,
(c) $H(t, \cdot)$ is upper semicontinuous, for almost all $t \in I$,
(d) there exists $h > 0$ such that $|H(t, x)| \leqslant h$, for all $(t, x) \in I \times \mathbb{R}^n$.

Let us define the Nemyckii operator $N_H : C(I, \mathbb{R}^n) \multimap L^1(I, \mathbb{R}^n)$ by

$$N_H(x) := \{f \in L^1(I, \mathbb{R}^n) \mid f(t) \in H(t, x(t)),\ \text{for almost all } t \in I\}. \tag{4}$$

Lemma 2. *Under assumptions* (a)–(d)*, the Nemyckii operator has the following properties*:

(i) *N_H has nonempty and convex values,*
(ii) *$\|N_H(x)\|_{L^1} \leqslant h\mu(I)$, for any $x \in C(I, \mathbb{R}^n)$, where μ stands for the Lebesgue measure,*
(iii) *N_H has norm-weak closed graph in the sense that if $x_n(t)$ converge to $x(t)$, for almost all $t \in I$, and $f_n \in N_H(x_n)$ converge to f, weakly in $L^1(I, \mathbb{R}^n)$, then $f \in N_H(x)$.*

Proof. ad (i). It is well known that the multivalued map $H(\cdot, x(\cdot))$ has a measurable selection, for any $x \in C(I, \mathbb{R}^n)$ (see, e.g., [5, Proposition 2.5] and the references therein). Since any such selection is integrable by (d), it belongs to $L^1(I, \mathbb{R}^n)$. Thus, N_H has nonempty values.

Let us show that N_H has convex values. Let $x \in C(I, \mathbb{R}^n)$ be arbitrary. Take $f_1, f_2 \in N_H(x)$ and choose $a, b \geqslant 0$ such that $a + b = 1$. Since $f_i \in N_H(x)$, for $i = 1, 2$, it follows that $f_i(t) \in H(t, x(t))$, for almost all $t \in I$. The convexity of values of H implies that $af_1(t) + bf_2(t) \in H(t, x(t))$, for almost all $t \in I$, and consequently, $af_1 + bf_2 \in N_H(x)$.

ad (ii). The $L^1(I, \mathbb{R}^n)$-boundedness of the values of N_H follows directly from (d).

ad (iii). This property is a particular case of [29, Theorem 3.1.2, p. 88]. $\square$

Let us now consider the operator differential inclusion

$$x' + \mathcal{G}(x) \in \mathcal{K}(x), \tag{5}$$

together with a constraint (e.g., a boundary or initial condition)

$$x \in S, \tag{6}$$

where S is a closed subset of $C(I, \mathbb{R}^n)$. Here,

$$\mathcal{G} : C\big(I, \mathbb{R}^n\big) \to L^1\big(I, \mathbb{R}^n\big)$$

is a continuous and bounded operator and

$$\mathcal{K} : C\big(I, \mathbb{R}^n\big) \multimap L^1\big(I, \mathbb{R}^n\big)$$

satisfies properties (i)–(iii) of Lemma 2.

It is well known that if x is an absolutely continuous function, then its derivative exists almost everywhere on I, it belongs to $L^1(I, \mathbb{R}^n)$ and satisfies

$$\int_s^t x'(\sigma)\, d\sigma = x(t) - x(s), \quad \text{on } I.$$

Therefore, by a solution to (5), (6), we shall understand an absolutely continuous function $x : I \to \mathbb{R}^n$, which satisfies (5) and (6), almost everywhere on I.

We shall now prove a lemma characterizing the solution operator of a fully linearized problem (5), (6).

Lemma 3. *Let*

- $\mathcal{G} : C(I, \mathbb{R}^n) \to L^1(I, \mathbb{R}^n)$ *be a continuous and bounded operator,*
- $\mathcal{K} : C(I, \mathbb{R}^n) \multimap L^1(I, \mathbb{R}^n)$ *satisfy properties* (i)–(iii) *of Lemma* 2*,*
- S *be a convex and closed subset of* $C(I, \mathbb{R}^n)$.

Consider a closed and bounded subset $Q \subset C(I, \mathbb{R}^n)$ *such that, for any* $q \in Q$*, the set of solutions to*

$$x' + \mathcal{G}(q) \in \mathcal{K}(q), \quad x \in S, \tag{7}$$

is nonempty. Denote by T the solution operator which takes any $q \in Q$ to the set of all solutions to (7) *and assume that $T(Q) \subset Q$. Then*

- *T has nonempty, compact and convex values,*
- *T is compact,*
- *T is upper semicontinuous.*

Proof. The operator T has nonempty values by the hypothesis. Let us prove the convexity of the values. Take $q \in Q$ arbitrary and $a, b \geqslant 0$ such that $a + b = 1$. Inclusion $x_i \in T(q)$, for $i = 1, 2$, is equivalent to $x_i' + \mathcal{G}(q) \in \mathcal{K}(q)$ and $x_i \in S$. Since $\mathcal{K}(q)$ is convex, we have

$$
\begin{gathered}
a\bigl(x_1' + \mathcal{G}(q)\bigr) + b\bigl(x_2' + \mathcal{G}(q)\bigr) \in \mathcal{K}(q), \\
ax_1' + bx_2' + \mathcal{G}(q) \in \mathcal{K}(q), \\
ax_1' + bx_2' \in T(q),
\end{gathered}
$$

which proves the convexity of the values of T.

Let us now show that $T(Q)$ is a relatively compact subset of $C(I, \mathbb{R}^n)$. Since $T(Q) \subset Q$ and Q is bounded, it is sufficient to show that $T(Q)$ is equicontinuous. The compactness of $T(Q)$ then follows from the Arzelà–Ascoli Theorem. Since Q and $\mathcal{G}$ are bounded, so is $\sup_{q \in Q}\{\|\mathcal{G}(q)\|_{L^1}\}$. Moreover, $\sup_{q \in Q}\{\|\mathcal{K}(q)\|_{L^1}\} < \infty$ because of item (ii) of Lemma 2. Altogether, $|x'|$ is uniformly bounded almost everywhere on I, and therefore $T(Q)$ is equicontinuous.

We shall now prove that T has a closed graph. This will imply (together with the compactness of T) that T has compact values and also that T is upper semicontinuous. Consider a sequence q_n which converges to q in $C(I, \mathbb{R}^n)$, and $x_n \in T(q_n)$ such that x_n converges to x in $C(I, \mathbb{R}^n)$. We want to show that $x \in T(q)$. The boundedness of Q together with the uniform boundedness of $|x_n'|$ (almost everywhere on I) gives according to [8, Theorem 4, p. 13] (cf. also [7]) a selected subsequence $\{x_k\} \subset \{x_n\}$ such that $x_k \to x$ uniformly on I, and $x_k' \to x'$ weakly in $L^1(I, \mathbb{R}^n)$. Thus, from the continuity of $\mathcal{G}$, we have $x_k' + \mathcal{G}(q_k) \to x' + \mathcal{G}(q)$, weakly in $L^1(I, \mathbb{R}^n)$, and according to item (iii) of Lemma 2, $x' + \mathcal{G}(q) \in \mathcal{K}(q)$, which means that $x \in T(q)$. So, the graph of T is closed, and consequently, T is upper semicontinuous with compact values. □

If Q is an ANR-space with a finitely generated Abelian fundamental group and $T : Q \multimap Q$ a compact upper semicontinuous map with R_δ-values, following the foregoing section, we can define the Nielsen number $N(T)$ which provides the lower estimate for the number of fixed points of T in Q. Moreover, $N(T)$ is a homotopy invariant in the sense that any $\widetilde{T} : Q \multimap Q$ (again an upper semicontinuous map with R_δ-values) which is compactly homotopic to T has also at least $N(T)$ fixed points in Q. Since each fixed point of T is a solution to (5) and (6), we can state the following theorem:

Theorem 1. *Let the assumptions of Lemma* 3 *be satisfied. Let Q be an ANR-space with a finitely generated Abelian fundamental group. Then problem* (5) *and* (6) *has at least $N(T)$ solutions in Q.*

Remark 2. As pointed out in Remark 1, instead of the fundamental group of Q being finitely generated and Abelian, one may assume Q to be compact. In the single-valued case of differential equations, none of these conditions needs to be satisfied.

Remark 3. Theorem 1 applies even if the solution operator T is not a self-map of the parameter space Q. More precisely, the conclusion of Theorem 1 holds, if $T : Q \multimap U$, where U is an open subset of $C(I, \mathbb{R}^n)$, is only *retractible* onto Q, namely if there exists a (continuous) retraction $s : U \to Q$ such that $p \in U \setminus Q$ with $s(p) = q$ implies that $p \notin T(q)$. In this case, we have $\mathrm{Fix}(T) = \mathrm{Fix}(s|_{T(Q)} \circ T)$ and, in view of Remark 1, the Nielsen number $N(s|_{T(Q)} \circ T)$ is well-defined for $s|_{T(Q)} \circ T : Q \multimap Q$. Moreover, $N(s|_{T(Q)} \circ T) \leqslant \#\mathrm{Fix}(T)$.

4. Nontrivial example

Consider the system of differential inclusions

$$x_1' + ax_1 \in u(Bx_2)\cos\varphi - u(Bx_1)\sin\varphi + \varphi H_1(t, x_1, x_2), \tag{8}$$

$$x_2' + ax_2 \in u(Bx_1)\cos\varphi + u(Bx_2)\sin\varphi + \varphi H_2(t, x_1, x_2). \tag{9}$$

Here, $(x_1, x_2) = x : [0, \omega] \to \mathbb{R}^2$ is the desired absolutely continuous solution, $a > 0$ is a given constant, $\varphi \in [0, \frac{\pi}{4}]$ is a homotopic parameter and $H : [0, \omega] \times \mathbb{R}^2 \multimap \mathbb{R}^2$ is a nonlinear perturbation satisfying items (a)–(d) of the foregoing section. Let $u : \mathbb{R} \to \mathbb{R}$ be a continuous bounded function the properties of which are to be specified later. Let $B : C([0, \omega], \mathbb{R}) \to C([0, \omega], \mathbb{R})$ be an operator defined as follows:

$$Bx_i(t) = x_i(0) + b\big(x_i(t) - x_i(0)\big),$$

for $i = 1, 2$, with some $b \in [0, 1]$. On constant functions, operator B becomes the identity; otherwise, it reduces the amplitude of a function by the factor b. Observe that if $q(t) = q(0) + \tilde{q}(t)$, where $|\tilde{q}(t)| \leqslant \varepsilon$, then $Bq(t) = q(0) + b\tilde{q}(t)$.

For $\varphi = \frac{\pi}{4}$, system (8), (9) takes the form

$$x_1' + ax_1 \in \frac{\sqrt{2}}{2}\big(u(Bx_2) - u(Bx_1)\big) + \frac{\pi}{4} H_1(t, x_1, x_2), \tag{10}$$

$$x_2' + ax_2 \in \frac{\sqrt{2}}{2}\big(u(Bx_1) + u(Bx_2)\big) + \frac{\pi}{4} H_2(t, x_1, x_2), \tag{11}$$

while for $\varphi = 0$, it simplifies to

$$x_1' + ax_1 = u(Bx_2), \tag{12}$$

$$x_2' + ax_2 = u(Bx_1). \tag{13}$$

We shall be looking for a periodic solution to problem (10), (11).

Let us define

$$\mathcal{G} : C\big([0, \omega], \mathbb{R}^2\big) \to C\big([0, \omega], \mathbb{R}^2\big) \subset L^1\big([0, \omega], \mathbb{R}^2\big)$$

by

$$\mathcal{G}(x) := \big(-ax_1 + u(Bx_2)\cos\varphi - u(Bx_1)\sin\varphi, -ax_2 + u(Bx_1)\cos\varphi + u(Bx_2)\sin\varphi\big).$$

Let

$$\mathcal{K}: C\big([0,\omega],\mathbb{R}^2\big) \multimap L^1\big([0,\omega],\mathbb{R}^2\big)$$

be the Nemyckii operator associated to H, i.e., let $\mathcal{K} := N_H$, where N_H was defined in (4) and $I = [0,\omega]$.

Problem (8), (9) can now be written in the form

$$x' + \mathcal{G}(x) \in \varphi\mathcal{K}(x)$$

and it is an easy observation that the operators $\mathcal{G}$ and $\varphi\mathcal{K}$ satisfy the assumptions of Lemma 3, independently of $\varphi \in [0, \frac{\pi}{4}]$.

Let us define the set $Q \subset C([0,\omega],\mathbb{R}^2)$ as follows. Function $q = (q_1, q_2)$ belongs to Q if the following conditions are satisfied:

(Q1) ω-periodicity: $q(0) = q(\omega)$,
(Q2) boundedness from above: $|q(t)| \leqslant R$, for all $t \in [0,\omega]$,
(Q3) boundedness from below: $|q(t)| \geqslant \delta$, for all $t \in [0,\omega]$,
(Q4) function q differs from its initial value at most by ε: $q(t) = q(0) + \tilde{q}(t)$, where $|\tilde{q}(t)| \leqslant \varepsilon$, for all $t \in [0,\omega]$.

Let us prove that Q is an ANR-space by showing that it is a neighbourhood retract in the Banach space of continuous ω-periodic functions $C_{\mathrm{per}} := \{p \in C([0,\omega],\mathbb{R}^2) \mid p(0) = p(\omega)\}$ (for this sufficiency, see [10]). Take a sufficiently small $\sigma > 0$ and consider the set $P \subset C_{\mathrm{per}}$ defined as follows. A function $p = (p_1, p_2)$ belongs to P if the following conditions are satisfied:

(P1) boundedness from above: $|p(t)| < R + \sigma$, for all $t \in [0,\omega]$,
(P2) boundedness from below: $|p(t)| > \delta - \sigma$, for all $t \in [0,\omega]$,
(P3) function p differs from its initial value by less than $\varepsilon + \sigma$: $p(t) = p(0) + \tilde{p}(t)$, where $|\tilde{p}(t)| < \varepsilon + \sigma$, for all $t \in [0,\omega]$.

One can readily check that the set P is an open neighbourhood of Q in C_{per}. Now, take $p \in P$ and consider a series of retractions. Firstly, retract the values of p which differ from $p(0)$ by more than ε onto a sphere of radius ε centered at $p(0)$. This retraction is defined in the radial direction with respect to point $p(0)$. Secondly, retract the values whose norm exceeds R onto a sphere or radius R centered at zero. Thirdly, retract the values whose norm is less that δ onto the sphere of radius δ centered at zero. These two retractions are also defined in the radial direction with respect to point 0. Observe that the composition of the three mappings is a continuous retraction of P onto Q.

By analogous arguments as above, one can show that the set of constant functions

$$\begin{aligned}\overline{Q} := \big\{(q_1,q_2) \in C\big([0,\omega],\mathbb{R}^2\big) \bigm| q_1(t) = \bar{q}_1,\, q_2(t) = \bar{q}_2,\ &\text{for all } t \in [0,\omega],\\ \text{and } \bar{q}_1^2 + \bar{q}_2^2 = \delta^2\big\}&\end{aligned}$$

is a deformation retract of Q. The fundamental group of $\overline{Q}$ is isomorphic to $\mathbb{Z}$, and subsequently, the fundamental group of Q must be also isomorphic to $\mathbb{Z}$.

Let $S := C_{\text{per}}$, i.e., let the set $S \subset C([0,\omega], \mathbb{R}^2)$ be defined to consist of functions which satisfy item (Q1). Observe that S is closed and convex.

Let us now specify the properties of function $u : \mathbb{R} \to \mathbb{R}$. Let the function u be bounded and continuous and such that

(U1) $|u(s)| \geqslant \delta + c_1$, for $s \in [\frac{\sqrt{2}\delta}{2}, R]$, with a suitable $c_1 > 0$,
(U2) $|u(s)| \leqslant R - c_2$, for $s \in [0, R]$, with a suitable $c_2 > 0$,
(U3) $|u(s_1) - u(s_2)| \leqslant L|s_1 - s_2|$, for all $s_1, s_2 \in [-R, R]$, with a suitable $L > 0$,
(U4) $u(-s) = -u(s)$, for all $s \in [-R, R]$.

We are now going to linearize system (8), (9) and show that, for any $\varphi \in [0, \frac{\pi}{4}]$, the solution operator, which takes $q \in Q$ to all the solutions x of the fully linearized problem, takes values in Q. Thus, we shall obtain a one-parameter family of operators $T_\varphi : Q \multimap Q$. Observe that, in view of Lemma 3 whose assumptions are satisfied, T_φ is a compact R_δ-mapping, for any $\varphi \in [0, \frac{\pi}{4}]$. The operator T_0 becomes simple and its Nielsen number can be computed explicitly by means of Lemma 1. It is easy to check that T_0 is homotopic to $T_{\pi/4}$ and, by means of Theorem 1, we shall establish the lower estimate for the number of fixed points of the operator $T_{\pi/4}$ which represent periodic solutions to (10), (11).

Let us now consider an arbitrary $q \in Q$. The fully linearized system (8), (9) takes the form

$$x_1' + ax_1 \in u(Bq_2)\cos\varphi - u(Bq_1)\sin\varphi + \varphi H_1(t, q_1, q_2), \tag{14}$$

$$x_2' + ax_2 \in u(Bq_1)\cos\varphi + u(Bq_2)\sin\varphi + \varphi H_2(t, q_1, q_2). \tag{15}$$

Any solution $x \in T_\varphi(q)$ can be written in the form

$$x(t) = \int_0^\omega G(t,s) f(s)\, ds, \tag{16}$$

where f is a selection from the right-hand side of inclusions (14), (15), namely,

$$f_1(s) := u(Bq_2)\cos\varphi - u(Bq_1)\sin\varphi + \varphi g_1, \tag{17}$$

$$f_2(s) := u(Bq_1)\cos\varphi + u(Bq_2)\sin\varphi + \varphi g_2, \tag{18}$$

with $g \in N_H(q)$. The function G in relation (16) stands for the Green function associated to the problem, which in this case takes the form

$$G(t,s) = \begin{cases} Ae^{a(s-t)}, & \text{for } 0 \leqslant s \leqslant t, \\ Be^{a(s-t)}, & \text{for } t \leqslant s \leqslant \omega, \end{cases} \tag{19}$$

$$\text{with } A = \frac{1}{1 - e^{-a\omega}} \text{ and } B = \frac{e^{-a\omega}}{1 - e^{-a\omega}}.$$

We shall now prove that, for any $\varphi \in [0, \frac{\pi}{4}]$, the set Q is invariant under the solution operator T_φ.

Lemma 4. *There exist constants L, b, h, δ, R, c_1, c_2 and ε such that, for any $\varphi \in [0, \frac{\pi}{4}]$, we have $T_\varphi(Q) \subset Q$.*

Proof. Take $q \in Q$ arbitrary. We shall prove that any $x \in T_\varphi(q)$ satisfies items (Q1)–(Q4) of the definition of the parameter set Q.

ad (Q1). The equality $x(0) = x(\omega)$ is guaranteed by (16) and (19).

ad (Q4). Let $q(t) = q(0) + \tilde{q}(t)$, where $|\tilde{q}(t)| \leqslant \varepsilon$, for all $t \in [0, \omega]$. Observe that $Bq(t) = q(0) + b\tilde{q}(t)$. For any $t \in [0, \omega]$, we have by (U3) the estimate

$$\left|u\bigl(q(0)\bigr) - u\bigl(Bq(t)\bigr)\right| = \left|u\bigl(q(0)\bigr) - u\bigl(q(0) + b\tilde{q}(t)\bigr)\right| \leqslant Lb\left|\tilde{q}(t)\right| \leqslant Lb\varepsilon. \tag{20}$$

Examining relations (17) and (18), we arrive at

$$f_i(s) = F_i + \tilde{f}_i(s), \tag{21}$$

where

$$F_1 := u\bigl(q_2(0)\bigr)\cos\varphi - u\bigl(q_1(0)\bigr)\sin\varphi, \qquad F_2 := u\bigl(q_1(0)\bigr)\cos\varphi + u\bigl(q_2(0)\bigr)\sin\varphi,$$

and the functions f_i satisfy the inequality

$$\left|\tilde{f}_i(s)\right| \leqslant \sqrt{2}Lb\varepsilon + \frac{\pi h}{4}, \tag{22}$$

for all $s \in [0, \omega]$. Estimate (22) can be shown as follows. From (20), we have $u(Bq_i(t)) = u(Bq_i(0)) + \tilde{u}_i(t)$, where $|\tilde{u}_i(t)| \leqslant Lb\varepsilon$. Thus,

$$\left|\tilde{f}_i(s)\right| \leqslant \left|\tilde{u}_i(s)\right|(\cos\varphi + \sin\varphi) + \varphi\left|g_i(s)\right| \leqslant \sqrt{2}Lb\varepsilon + \frac{\pi h}{4}.$$

Substituting expression (21) into (16), we obtain

$$x_i(t) = \frac{F_i}{a} + \tilde{x}_i(t)$$
$$\text{with } \left|\tilde{x}_i(t)\right| \leqslant \frac{\sqrt{2}Lb\varepsilon}{a} + \frac{\pi h}{4a}.$$

We can now take L, b and h small enough to satisfy

$$\frac{\sqrt{2}Lb\varepsilon}{a} + \frac{\pi h}{4a} \leqslant \frac{\varepsilon}{2}. \tag{23}$$

This ensures that any $x \in T_\varphi(q)$ differs from a constant function by less than $\frac{\varepsilon}{2}$, and consequently, it differs from its initial value by less than ε.

ad (Q2) and (Q3). Since we have just proved that T_φ takes functions differing from its initial value by less than ε to functions with the same property, let us deal with constant functions first. Let us consider a constant function $\bar{q} \in Q$ and take $x \in T_\varphi(\bar{q})$ arbitrary. Clearly, x differs from the constant function $\frac{F}{a}$ by less than $\frac{\pi h}{4a}$. The operator which takes $\bar{q}$ to $\frac{F}{a}$ is a composition of

- reflection $(\bar{q}_1, \bar{q}_2) \to (\bar{q}_2, \bar{q}_1)$,
- rescaling $(\bar{q}_1, \bar{q}_2) \to \frac{1}{a}(u(\bar{q}_1), u(\bar{q}_2))$,
- rotation $(\bar{q}_1, \bar{q}_2) \to (\bar{q}_1 \cos\varphi - \bar{q}_2 \sin\varphi, \bar{q}_2 \cos\varphi + \bar{q}_1 \sin\varphi)$ by the angle φ in the anti-clockwise direction.

The rescaling part of the composition together with properties (U1) and (U2) of the function u guarantee that constants R, δ and c_1, c_2 exist such that T_φ takes constant functions satisfying (Q2) and (Q3) to functions that again satisfy (Q2) and (Q3). Since functions in Q differ from their initial values by less than ε, we only need to guarantee that

$$\frac{\sqrt{2}(R - c_2)}{a} + \varepsilon \leqslant R, \tag{24}$$

$$\frac{\delta + c_1}{a} - \varepsilon \geqslant \delta. \tag{25}$$

Indeed, take any $\bar{q} \in Q$ and observe that

$$\begin{aligned}
\bar{q}_1^2 + \bar{q}_2^2 \leqslant R^2 \quad &\Rightarrow \quad |\bar{q}_1| \leqslant R \quad \text{and} \quad |\bar{q}_2| \leqslant R \\
&\Rightarrow \quad \bigl|u(\bar{q}_1)\bigr| \leqslant R - c_2 \quad \text{and} \quad \bigl|u(\bar{q}_2)\bigr| \leqslant R - c_2 \\
&\Rightarrow \quad u^2(\bar{q}_1) + u^2(\bar{q}_2) \leqslant 2(R - c_2)^2 \\
&\Rightarrow \quad |F| \leqslant \sqrt{2}(R - c_2) \\
&\Rightarrow \quad \bigl|x(t)\bigr| \leqslant \frac{\sqrt{2}(R - c_2)}{a} + \varepsilon, \quad \text{for all } t \in [0, \omega].
\end{aligned}$$

Similarly,

$$\begin{aligned}
\bar{q}_1^2 + \bar{q}_2^2 \geqslant \delta^2 \quad &\Rightarrow \quad |\bar{q}_1| \geqslant \frac{\sqrt{2}\delta}{2} \quad \text{or} \quad |\bar{q}_2| \geqslant \frac{\sqrt{2}\delta}{2} \\
&\Rightarrow \quad \bigl|u(\bar{q}_1)\bigr| \geqslant \delta + c_1 \quad \text{or} \quad \bigl|u(\bar{q}_2)\bigr| \geqslant \delta + c_1 \\
&\Rightarrow \quad u^2(\bar{q}_1) + u^2(\bar{q}_2) \geqslant (\delta + c_1)^2 \\
&\Rightarrow \quad |F| \geqslant \delta + c_1 \\
&\Rightarrow \quad \bigl|x(t)\bigr| \geqslant \frac{\delta + c_1}{a} - \varepsilon, \quad \text{for all } t \in [0, \omega]. \qquad \square
\end{aligned}$$

To get a clear idea of what is happening, let us present an example of such a situation, where all the above inequalities are satisfied. Put

$$u(s) := \begin{cases} \sqrt[3]{s} + 2, & \text{for } s \geqslant 1, \\ 3s, & \text{for } -1 \leqslant s \leqslant 1, \\ \sqrt[3]{s} - 2, & \text{for } s \leqslant -1, \end{cases}$$

and set $\delta = 1$, $R = 10$ and $a = 1$. Observe that (U1) holds with $c_1 = 1$ and (U2) holds with $c_2 = 5$. We can thus take $\varepsilon = 1$, and see that (24), (25) holds. Since $L \leqslant 3$, we need $b \leqslant \frac{1}{12\sqrt{2}}$ and

$h \leqslant \frac{1}{\pi}$. Then estimate (23) holds as well. Figure 1 shows how the operator $T_{\pi/4}$ treats constant functions on Q. The figure reveals that there are no easily detectable subdomains of Q which would be separately invariant under $T_{\pi/4}$. The subinvariance of easily detectable subdomains was the major drawback of the previously given examples, as mentioned in the introduction. Therefore, we have no obvious means of avoiding the usage of the Nielsen theory (e.g., standard fixed point theorems or the index additivity property) to obtain a multiplicity result.

Now, let us return to the general setting. According to Lemma 3, system (14), (15) possesses, for any $q \in Q$, a compact and convex set of solutions $T_\varphi(q)$ and the operator $T_\varphi : Q \multimap Q$ is upper semicontinuous. Note that (16) together with the properties of H ensure that, for any $q \in Q$ and any $\varphi \in [0, \frac{\pi}{4}]$, the set $T_\varphi(q)$ is nonempty. The inclusion $T(Q) \subset Q$ is shown in Lemma 4, so the assumptions of Lemma 3 are satisfied.

According to Theorem 1, system (10), (11) admits at least $N(T_{\pi/4})$ periodic solutions. Since $T_{\pi/4}$ is homotopic to T_0, through a homotopy with compact and convex values which do not leave Q, we have $N(T_{\pi/4}) = N(T_0)$. Note that T_0 is single-valued.

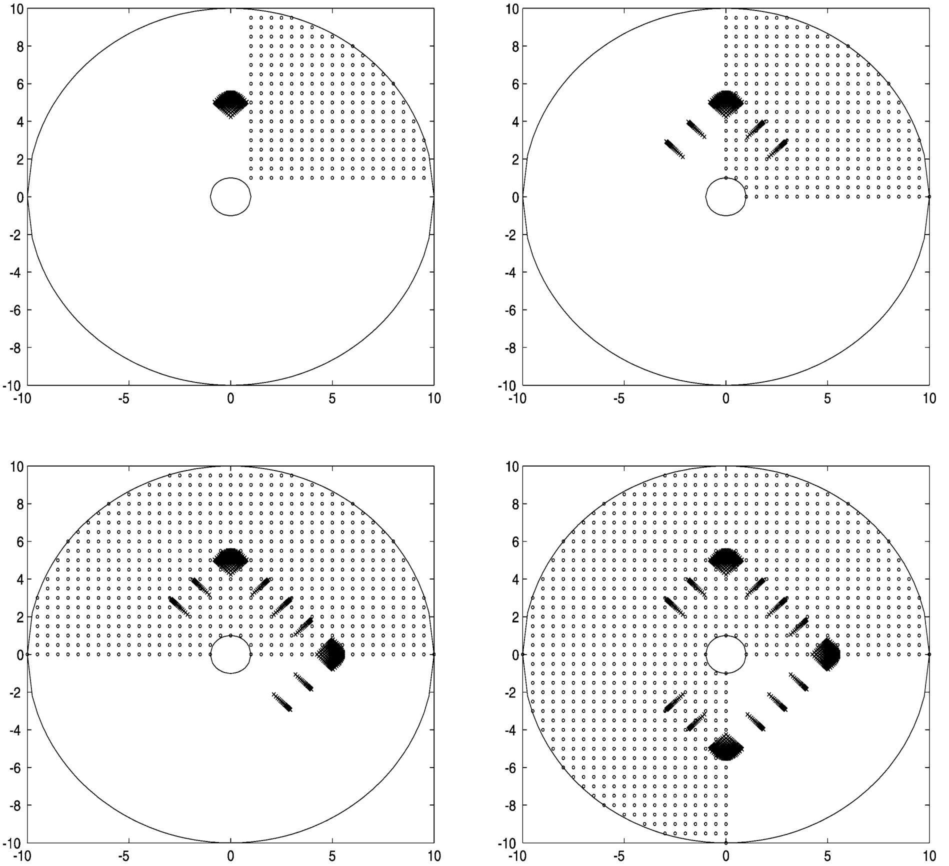

Fig. 1. Behaviour of $T_{\pi/4}$ on constant functions on Q, for $R = 10$, $\delta = 1$ and $a = 1$. Rectangular grids of points represent constant functions $q \in Q$, the accumulated sets of points represent their images under $T_{\pi/4}$. For simplicity, we take here $H \equiv 0$, so that the images of constant functions become constant again. No easily detectable regions of the domain are subinvariant.

Let us further consider the mapping $r : Q \to Q$ which sends a function $q \in Q$ to its initial value $q(0)$. Let us define the homotopy $T^{\mu} : [0,1] \times Q \to Q$ by

$$T^{\mu}(q) := \mu T_0(q) + (1-\mu) T_0\big(r(q)\big).$$

This homotopy guarantees that $N(T_0) = N(T_0 \circ r)$. We can thus restrict ourselves to the computation of $N(T_0 \circ r)$. Let us denote by $\overline{Q}$ the subset of Q consisting of constant functions. Since $T_0 \circ r : Q \to \overline{Q}$, all the fixed points of $T_0 \circ r$ have to belong to $\overline{Q}$. Let us therefore deal with the restriction

$$L := T_0 \circ r|_{\overline{Q}} = T_0|_{\overline{Q}} : \overline{Q} \to \overline{Q}$$

which can be explicitly written in the form

$$L(\bar{q}_1, \bar{q}_2) := \frac{1}{a}\big(u\big(q_2(0)\big), u\big(q_1(0)\big)\big).$$

Consider the closed convex sets $K_1, K_2 \subset \mathbb{R}^2$ defined by

$$K_1 := \big([\delta, R] \times [\delta, R]\big) \cap \overline{Q} \quad \text{and} \quad K_2 := \big([-R, -\delta] \times [-R, -\delta]\big) \cap \overline{Q}.$$

Inequalities (24) and (25) together with (U1), (U2) and (U4) guarantee that $L(K_1) \subset K_1$ and $L(K_2) \subset K_2$. Since u is continuous by (U3), the standard Brouwer Fixed Point Theorem ensures the existence of at least one fixed point in each K_i. Since both the sets are convex, all fixed points in K_1 lie in the same Nielsen class and all fixed points in K_2 lie in the same Nielsen class. Now, choose two fixed points $k_1 \in K_1$ and $k_2 \in K_2$ and choose an arbitrary path C connecting k_1 and k_2 in $\overline{Q}$. The image of this path under the mapping L, $L \circ C$, is clearly not homotopic with the original path C (see Fig. 2). This is ensured by the reflection part of L. Thus, points k_1 and k_2

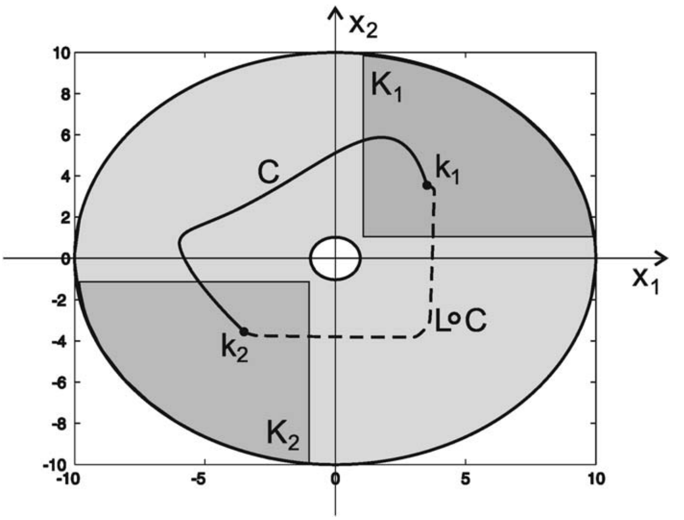

Fig. 2. The set $\overline{Q}$ with L-subinvariant domains K_1 and K_2 which contain two fixed points $k_1 \in K_1$ and $k_2 \in K_2$. Path C connecting k_1 and k_2 is not homotopic (keeping the end points fixed) to path $L \circ C$. This demonstrates that the two fixed points belong to different Nielsen classes.

belong to different Nielsen classes. Moreover, both the Nielsen classes are essential, because the related fixed point indices are equal to 1 (cf. [24]), and therefore $N(L) = 2$. Applying the reduction property of the Nielsen number in Lemma 1, we arrive at

$$N(T_{\frac{\pi}{4}}) = N(T_0) = N(T_0 \circ r) = N(L) = 2$$

which jointly with Theorem 1 guarantees the existence of at least two ω-periodic solutions of problem (10), (11) that are located in the set Q.

We are in the position to formulate the multiplicity criterium for ω-periodic solutions to system (10), (11).

Theorem 2. *Let inequalities* (23)–(25) *be satisfied. Then there exists at least two ω-periodic solutions of system* (10), (11) *that are located in the set Q.*

In fact, it can be shown, by the additivity property of the related fixed point index, that system (10), (11) admits at least three ω-periodic solutions, provided the sharp inequality holds in (23).

Corollary 1. *Let inequalities* (24), (25) *be satisfied and let* (23) *hold with the sharp inequality. Then system* (10), (11) *admits at least three ω-periodic solutions.*

Proof. Consider the set $Q_1 \subset C([0,\omega], \mathbb{R}^2)$ defined by relations (Q1) and (Q2) with the sharp inequality in (Q2). Observe that Q_1 is open in the space of continuous ω-periodic functions on $[0,\omega]$. Consider further $\mathring{Q}$ (the interior of the set Q relative to the space of continuous and ω-periodic functions on $[0,\omega]$) and observe that a function q belongs to $\mathring{Q}$ if and only if it satisfies (Q1) and items (Q2)–(Q4) with the sharp inequalities in (Q2), (Q3), (Q4). Assume that (23) holds with the sharp inequality. Then the same argument as above shows that $\mathring{Q}$ is invariant under the solution operator $T_{\pi/4}$. By the homotopy and reduction properties of the related well-defined fixed point index, (cf., e.g., [4,7]), we arrive at

$$\operatorname{ind}(T_{\frac{\pi}{4}}|_{\mathring{Q}}, \mathring{Q}) = \operatorname{ind}(T_0|_{\mathring{Q}}, \mathring{Q}) = \operatorname{ind}(L, \overline{Q}) = 2.$$

Let us now show that the set Q_1 is also invariant under the operator $T_{\pi/4}$. Take arbitrary $q \in Q_1$ and recall that $T_{\pi/4}(q)$ is given by (16)–(19). The ω-periodicity of $T_{\pi/4}(q)$ is obvious, it remains to show that $|T_{\pi/4}(q)(t)| < R$, for any $t \in [0,\omega]$. Take $t \in [0,\omega]$ and observe that

$$\begin{aligned}
\left|T_{\frac{\pi}{4}}(q)(t)\right| &\leqslant \int_0^{\omega} \left|G(t,s)f(s)\right| ds \leqslant \max_{s\in[0,\omega]} \left|f(s)\right| \int_0^{\omega} G(t,s)\, ds \\
&\leqslant \frac{\sqrt{2}(R-c_2)}{a} + \varepsilon \quad \bigl(\text{by (U2)}\bigr) \\
&< R \quad \bigl(\text{by (23) with the sharp inequality and (24)}\bigr).
\end{aligned}$$

Since Q_1 is an AR-space, the related fixed point index satisfies

$$\operatorname{ind}(T_{\frac{\pi}{4}}, Q_1) = 1$$

(see [4,7]).

By the additivity property of the fixed point index, i.e.,

$$\mathrm{ind}(T_{\frac{\pi}{4}}, Q_1) = \mathrm{ind}(T_{\frac{\pi}{4}}|_{\mathring{Q}}, \mathring{Q}) + \mathrm{ind}(T_{\frac{\pi}{4}}|_{Q_1\setminus Q}, Q_1\setminus Q),$$

we obtain that

$$\mathrm{ind}(T_{\frac{\pi}{4}}|_{Q_1\setminus Q}, Q_1\setminus Q) = -1.$$

Thus, by the existence property, the set $Q_1\setminus Q$ contains the third fixed point which represents the third ω-periodic solution to (10), (11). □

Example 1. The system

$$x_1'(t) + x_1(t) = \frac{\sqrt{2}}{2}\big[u\big(x_2(0) + b\big(x_2(t) - x_2(0)\big)\big) - u\big(x_1(0) + b\big(x_1(t) - x_1(0)\big)\big)\big] + \frac{1}{4}\sin t, \tag{26}$$

$$x_2'(t) + x_2(t) = \frac{\sqrt{2}}{2}\big[u\big(x_1(0) + b\big(x_1(t) - x_1(0)\big)\big) + u\big(x_2(0) + b\big(x_2(t) - x_2(0)\big)\big)\big] + \frac{1}{4}\cos t, \tag{27}$$

e.g., with $b = \frac{1}{17}$ and

$$u(s) := \begin{cases} \sqrt[3]{s} + 2, & \text{for } s \geqslant 1, \\ 3s, & \text{for } -1 \leqslant s \leqslant 1, \\ \sqrt[3]{s} - 2, & \text{for } s \leqslant -1, \end{cases}$$

admits according to Corollary 1 at least three 2π-periodic solutions. On the other hand, the related phase-portraits in Figs. 3 and 4 reflect the complexity, which justifies the application of the Nielsen theory, but not a preferable reference to the proved periodic solutions. Rather curiously, if parameter b is greater than it is allowed by inequality (23) (e.g., $b = 0.95$), then two orbitally stable 2π-periodic solutions can easily be detected, while a possible third one (expected near the origin of the coordinate system) is not visible because of its instability.

5. Concluding remarks

Remark 4. Putting $\mathcal{G}: C(I, \mathbb{R}^n) \to L^1(I, \mathbb{R}^n)$ such that $\mathcal{G}(x) := A(t)x$, where $A: I \to \mathbb{R}^{n^2}$ is a continuous $(n \times n)$ matrix, and $\mathcal{K}: C(I, \mathbb{R}^n) \multimap L^1(I, \mathbb{R}^n)$ such that

$$\mathcal{K}(x) := \big\{ f \in L^1\big(I, \mathbb{R}^n\big) \bigm| f(t) \in F\big(t, x(t)\big), \text{ for almost all } t \in I \big\},$$

where $F: I \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping, jointly with

$$S := \big\{ x \in C\big(I, \mathbb{R}^n\big) \bigm| Lx = \Theta \big\}, \quad \Theta \in \mathbb{R}^n,$$

where $L: C(I, \mathbb{R}^n) \to \mathbb{R}^n$ is a linear operator such that the homogeneous problem

$$x' + A(t)x = 0, \quad Lx = 0,$$

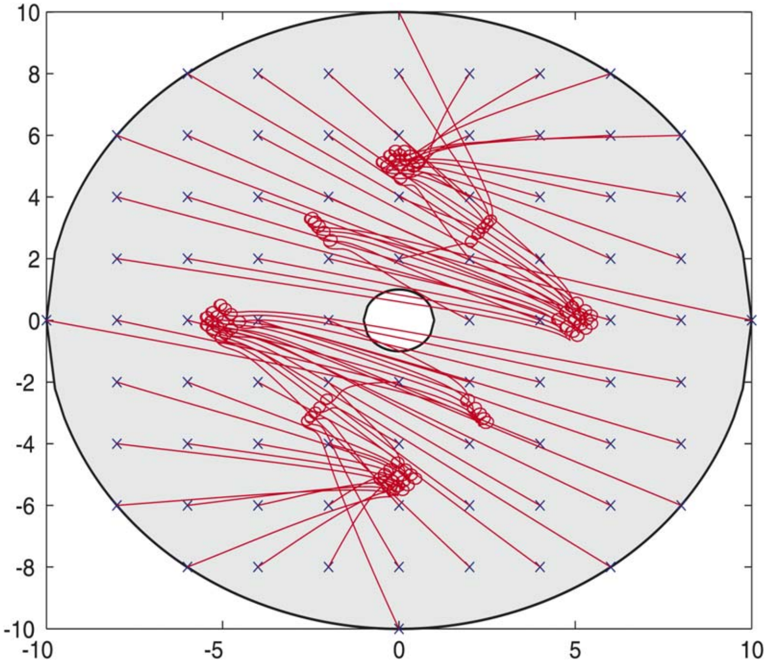

Fig. 3. Phase-portrait of system (26), (27) outside the neighbourhood of the origin of the coordinate system. The crosses indicate the initial values of the trajectories.

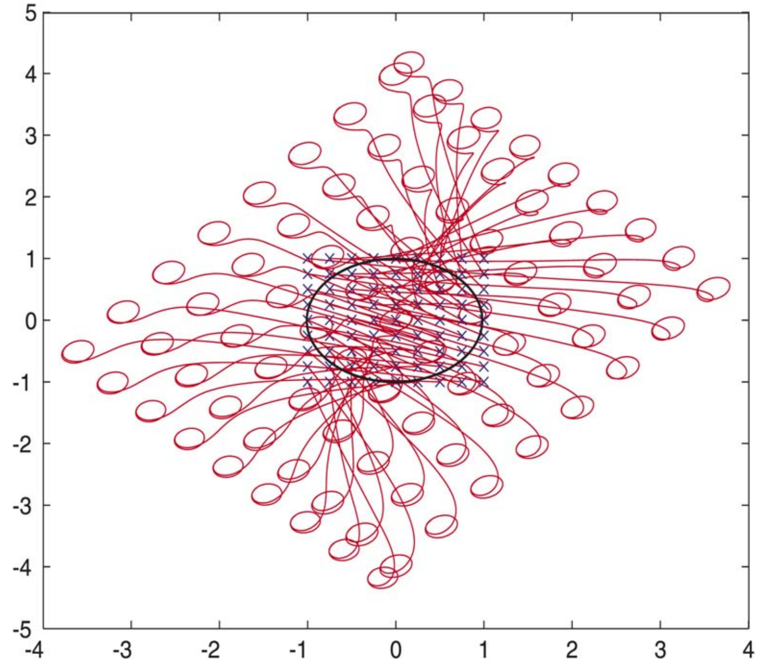

Fig. 4. Phase-portrait of system (26), (27) near the origin of the coordinate system. The bold circle in the middle indicates the inner boundary of the set $\overline{Q}$.

has only the trivial solution on I, we have a particular case of the general setting treated in [1] (cf. [3,5,7]).

Remark 5. In [5] (and, in the single-valued case, in [23]), we found sufficient conditions for the existence of at least three ω-periodic solutions for a planar integro-differential system. This system can be obtained from the general operator setting by putting $\mathcal{G} : C([0,\omega],\mathbb{R}^2) \to L^1([0,\omega],\mathbb{R}^2)$ such that

$$\mathcal{G}(x) := \left(ax_1 - \frac{\sqrt{2}}{2}\sqrt[3]{p_2(x)} + \frac{\sqrt{2}}{2}\sqrt[3]{p_1(x)}, ax_2 - \frac{\sqrt{2}}{2}\sqrt[3]{p_1(x)} - \frac{\sqrt{2}}{2}\sqrt[3]{p_2(x)}\right),$$

with $a > 0$ and $x = (x_1, x_2)$, where

$$p_i(x) = \frac{1}{\omega}\int_0^\omega x_i(s)\,ds - B\left(\frac{1}{\omega}\int_0^\omega x_i(s)\,ds - x_i\right),$$

with $B > 0$, and defining $\mathcal{K} : C([0,\omega],\mathbb{R}^2) \to L^1([0,\omega],\mathbb{R}^2)$ by

$$\mathcal{K}(x) := \bigl\{ f \in L^1\bigl([0,\omega,\mathbb{R}^2]\bigr) \bigm| f(t) \in e\bigl(t, x(t)\bigr), \text{ for almost all } t \in [0,\omega] \bigr\},$$

where $e : [0,\omega] \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping, jointly with

$$S := \bigl\{ x \in C\bigl([0,\omega],\mathbb{R}^2\bigr) \bigm| x(0) = x(\omega) \bigr\}.$$

Remark 6. We can get a large variety of nontrivial examples of application of the Nielsen theory to operator differential inclusions. For instance, perturbation H in system (10), (11) can be replaced by an operator depending also on the solution $x(\cdot)$.

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